Complexité et minimalité pour les substitutions et les mots lisses

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Soutenance de thèse

Sous la supervision de Nicolas Bédaride et Étienne Moutot

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Words and symbols

dbebfjcgfdfijhjdcdiegcgeddidchjfaciiebjhbgjdjjdhfbafi . . .

Words and symbols

$$dbebfjcgfdfijhjdcdiegcgeddidchjfaciiebjhbgjdjjdhfbafi\dots$$

$$\updownarrow$$

$$31415926535897932384626433832795028841971693993751058\dots$$

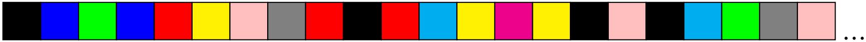
Words and symbols

dbebfjcgfdfijhjdcdiegcgeddidchjfaciiebjhbgjdjjdhfbafi ...

↕

31415926535897932384626433832795028841971693993751058...

↕



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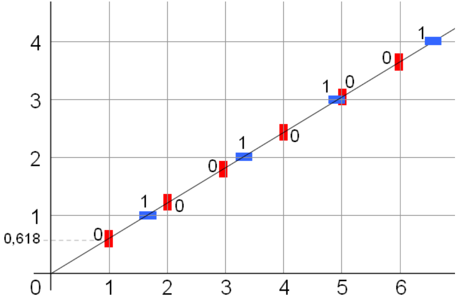
↕

31415926535897932384626433832795028841971693993751058...

↕

...



Part 1: Substitutions

Substitutions

$\sigma : \mathcal{A} \longrightarrow \mathcal{A}^* \; 0 \mapsto 01, 1 \mapsto 10$

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0

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$\underline{0}1$

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$\searrow$

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0110100110010110100101100110100110010110011010...

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$0110100110010110100101100110100110010110011010 \ldots = \sigma^{\infty}(0)$

- **Purely morphic sequence**

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01

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...

0110100110010110100101100110100110010110011010... $= \sigma^\infty(0)$

- **Purely morphic sequence**
- $\sigma^\infty(0)$ is **cube-free**

Substitutions

On plie du papier !

Substitutions

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paper-folding $= \wedge \wedge \vee \wedge \wedge \vee \vee \wedge \wedge \wedge \vee \vee \wedge \vee \vee \wedge \wedge \wedge \vee \wedge \wedge \vee \vee \vee \wedge \wedge \vee \vee \ldots$

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$\sigma : a \mapsto ab, b \mapsto cb, c \mapsto ad, d \mapsto cd$
$\sigma^{\infty}(a) = abcbadcbabcdadcbabcbadcdabcdadcbabcbadcbabcdadcdab\ldots$

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On plie du papier !

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- **Morphic sequence**

$x =$ abcbadcbabcdadcbabcbadcdabcdadcbabcbadcbabcdadcdabcbadcd...
Complexity $p_x(n) := |\mathscr{L}(x) \cap \mathcal{A}^n|$

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Complexity of factors

$x = \underline{a}b\underline{c}ba\underline{d}c\underline{b}abcdadcbabcbadcdabcdadcbabcbadcbabcdadcdabcbadcd\ldots$

Complexity $p_x(n) := |\mathscr{L}(x) \cap \mathcal{A}^n|$ $\quad p_x(1) = 4$

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Complexity $p_x(n) := |\mathscr{L}(x) \cap \mathcal{A}^n|$ $\quad p_x(2) = 8$

$x =$ a$\underline{\text{bcb}}$a$\underline{\text{dcb}}$a$\underline{\text{bcd}}$a$\underline{\text{dcb}}$abc$\underline{\text{bad}}$c$\underline{\text{dab}}$c$\underline{\text{dad}}$c$\underline{\text{bab}}$cb$\underline{\text{adc}}$b$\underline{\text{abc}}$dad$\underline{\text{cda}}$b$\underline{\text{cba}}$dcd...

Complexity $p_x(n) := |\mathscr{L}(x) \cap \mathcal{A}^n|$ $\quad p_x(3) = 12$

$x =$ abcbadcbabcdadcbabcbadcdabcdadcbabcbadcbabcdadcdabcbadcd...

Complexity $p_x(n) := |\mathscr{L}(x) \cap \mathcal{A}^n|$ $\quad$ $p_x(n) = 4n$

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- $1 \leq p_x(n) \leq |\mathcal{A}|^n$

Theorem [Morse, Hedlund 1938]

A sequence x is eventually periodic iff $p_x(n)$ is bounded.

Theorem *(D0L-systems)* [Ehrenfeucht, Lee, Rozenberg 1975-1983]

$p_x(n) = \mathcal{O}(n^2)$.

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Theorem *(Morphic)* [Devyatov 2015]

$p_x(n) = \Theta(n^{1+1/k})$ for $k \in \mathbb{N}^*$ or $\mathcal{O}(n \log n)$.

Proposition [H. 2024]

The classification of purely morphic sequences is effective.

- Towards a simpler proof of Devyatov's theorem

Uniform recurrence/minimality

Combinatorial POV:

Definition

A sequence $x \in \mathcal{A}^{\mathbb{N}}$ is **uniformly recurrent (UR)** if all its factors **appear with bounded gaps**.

Uniform recurrence/minimality

Dynamical POV:

Subshift : $X \subseteq \mathcal{A}^{\mathbb{Z}}$

- s.t. $S(X) = X$
- closed for the usual topology

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Definition/Proposition

A subshift X is **minimal** if, equivalently:

1. X has no subsystem.
2. $\forall x \in X, \overline{\mathscr{O}(x)} = X$.
3. Every word of $\mathscr{L}(X)$ occurs with bounded gaps in X.

Substitution subshifts

substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$

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language $\mathscr{L}(\sigma) := \{u \in \mathcal{A}^* \mid \exists a \in \mathcal{A}, \exists n \geq 0, u \sqsubset \sigma^n(a)\}$

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$\sigma : 0 \mapsto 0, 1 \mapsto 010$

Substitution subshifts

$$\textbf{substitution } \sigma : \mathcal{A} \to \mathcal{A}^*$$

$$\downarrow$$

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- $X_\sigma := X(\ldots 0001000 \ldots) = \{S^n(\ldots 0001000 \ldots) \mid n \in \mathbb{Z}\} \cup \{\ldots 000 \ldots\}$

Minimality

Folklore

σ **primitive** $\implies X_\sigma$ **minimal**.

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σ is **primitive** if $\exists n \geq 1$ s.t. $\forall a, b \in \mathcal{A}$, $a \sqsubset \sigma^n(b)$.

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Theorem [Shimomura 2019]

The **minimal** subshifts are those generated by **tame** and ℓ-**primitive** substitutions.

Minimality and beyond

Non-minimal case ▶ subsystems

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X_σ can be **essentially minimal** and not minimal

Non-minimal case ▶ subsystems

X_σ can be **essentially minimal** and not minimal $\overline{\mathscr{O}(\ldots0001000\ldots)}$

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Proposition [Béal, Perrin, Restivo 2024]

Every substitution subshift is **quasi-minimal**.

Non-minimal case ▶ subsystems

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[Byszewski, Konieczny, Krawczyk 2021] for **growing** substitutions

An important distinction

Definition

- $\mathcal{B}$:= set of **bounded** letters: $(|\sigma^n(a)|)_{n\geq 0}$ bounded
- $\mathcal{C}$:= set of **growing** letters: $|\sigma^n(a)| \xrightarrow[n\to\infty]{} \infty$

Theorem *(Wild minimal components)* [H. 2025]

The minimal components of X_σ **in** $\mathcal{B}^{\mathbb{Z}}$ are computable **periodic orbits**.

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- $\{\ldots 11111 \ldots\} \in X_\sigma$

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The minimal components of X_σ **not in** $\mathcal{B}^{\mathbb{Z}}$ are the X_τ where τ is a **main sub-substitution** of σ.

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$$\sigma : 0 \mapsto 0102, 1 \mapsto 11, 2 \mapsto 23, 3 \mapsto 32$$

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The minimal components of X_σ **not in** $\mathcal{B}^{\mathbb{Z}}$ are the X_τ where τ is a **main sub-substitution** of σ.

$$\sigma : 0 \mapsto 0102, 1 \mapsto 11, 2 \mapsto 23, 3 \mapsto 32$$

- $\tau = \sigma|_{\{1\}} : 1 \mapsto 11$
- $\tau' = \sigma|_{\{2,3\}} : 2 \mapsto 23, 3 \mapsto 32$

Corollary *(Counting minimal components)* [H. 2025]

$MC(\sigma) :=$ number of minimal components of X_σ.

- If $|\mathcal{B}| = 0$, then $MC(\sigma) \leq |\mathcal{C}| = |\mathcal{A}|$.
- If $|\mathcal{B}| = 1$, then $MC(\sigma) \leq |\mathcal{C}| = |\mathcal{A}| - 1$.
- If $|\mathcal{B}| \geq 2$, then $MC(\sigma) \leq 2|\mathcal{C}| \leq 2|\mathcal{A}| - 4$.

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Part 2: Smooth sequences

Skew/staggered substitutions

$\sigma_0 : 1 \mapsto 2, 2 \mapsto 22$ $\sigma_1 : 1 \mapsto 1, 2 \mapsto 11$

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2

22

$22\underline{1}1$

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$2211\overline{2}$

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2

22

$221\underline{1}$

$\searrow \sigma_1$

$22112\overline{1}$

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2

22

2211

221121

...

22112122122112112212112122112112122122112122121121122122...

The Oldenburger-Kolakoski sequence

Definition [Oldenburger 1939] [Kolakoski 1965]

$$\kappa = \underbrace{2\,2}_{2}\ \underbrace{1\,1}_{2}\ \underbrace{2}_{1}\ \underbrace{1}_{1}\ \underbrace{2\,2}_{2}\ \underbrace{1}_{1}\ \underbrace{22}_{2}\ \underbrace{11}_{2}\ \underbrace{2}_{1}\ \underbrace{11}_{2}\ \underbrace{22}_{2}\ \underbrace{1}_{1}\ \underbrace{2}_{1}\ldots$$

fixed point of the **run-length encoding**.

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- κ is registered OEIS000002

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Theorem

- κ is **not eventually periodic** [Oldenburger 1939]
- κ is **not purely morphic** [Carpi 1993] [Lepistö 1993]
- κ is **cube-free** [Carpi 1994]

The Oldenburger-Kolakoski sequence

Conjectures

- κ is **not morphic**
- $\mathscr{L}(\kappa) = \mathcal{C}_f^\infty$ [Dekking 1981]
- κ is **UR** but not **LR**
- $p_\kappa(n) = p_{\mathcal{C}_f^\infty}(n) = \Theta(n^\rho)$, $\rho = \frac{\log(3)}{\log(3/2)} \approx 2.70951$ [Dekking 1981]
- κ has **uniform frequencies** (**uniquely ergodic**) [Dekking 2001]
- κ has **equal letter frequencies** [Keane 1991]

Smooth sequences

Definition

Over $\mathcal{A}$, a sequence of the form

$$a_0^{p_0} a_1^{p_1} a_2^{p_2} a_3^{p_3} \ldots$$

where $a_{i+1} = \overline{a_i}$ and $p_i \in \mathcal{A}$ is **derivable**, and we write $x \in \mathcal{C}^1$.

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The **derivative** is the map

$$\begin{aligned} \mathcal{D} \ : \qquad \mathcal{C}^1 &\longrightarrow \mathcal{A}^{\mathbb{N}} \\ a_0^{p_0} a_1^{p_1} a_2^{p_2} a_3^{p_3} \dots &\longmapsto p_0 p_1 p_2 p_3 \dots \end{aligned}$$

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$$\mathcal{D} \; : \quad \mathcal{C}^1 \longrightarrow \mathcal{A}^{\mathbb{N}}$$
$$a_0^{p_0} a_1^{p_1} a_2^{p_2} a_3^{p_3} \dots \longmapsto p_0 p_1 p_2 p_3 \dots$$

$\mathcal{D}((21)^\omega) = 1^\omega$ and $\mathcal{D}((221)^\omega) = (21)^\omega$

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n-times derivable sequences:

$$\mathcal{C}^n := \mathcal{D}^{-n}\left(\mathcal{A}^{\mathbb{N}}\right)$$

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Infinitely derivable sequences:

$$\mathcal{C}^\infty := \bigcap_{n \geq 0} \mathcal{C}^n$$

Smooth sequences

Definition

n-times derivable sequences:

$$\mathcal{C}^n := \mathcal{D}^{-n}\left(\mathcal{A}^{\mathbb{N}}\right)$$

Infinitely derivable sequences:

$$\mathcal{C}^\infty := \bigcap_{n \geq 0} \mathcal{C}^n$$

a.k.a. **smooth sequences**

Smooth sequences

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$$\mathcal{C}^n := \mathcal{D}^{-n}\left(\mathcal{A}^{\mathbb{N}}\right)$$

Infinitely derivable sequences:

$$\mathcal{C}^\infty := \bigcap_{n \geq 0} \mathcal{C}^n$$

a.k.a. **smooth sequences**

- No smooth sequence is eventually periodic

Smooth sequences

$x = $ 2212112112212211212212211211212212112212211212212211...

Smooth sequences

$$
\begin{aligned}
x &= 221211211221221121221221121121221211221221121221211\ldots \\
\mathcal{D}(x) &= 211212212211212212112112212211212212211211212212112\ldots \\
\mathcal{D}^2(x) &= 121121221121121221221121221211211221221211221221212\ldots \\
\mathcal{D}^3(x) &= 112112212112122112112122121122122112122122112112122 1\ldots \\
\mathcal{D}^4(x) &= 212211211221211211221221121221211211221211212211211 2\ldots \\
\mathcal{D}^5(x) &= 112212211212212211211212211211221211212212211211212 2\ldots \\
\mathcal{D}^6(x) &= 221221121221211221221121121221211221221211211221211 2\ldots \\
&\ldots
\end{aligned}
$$

$$
\begin{aligned}
x &= \color{orange}{2}\color{black}{212112112212211212212211211212212112212211212212211}\ldots \\
\mathcal{D}(x) &= \color{orange}{2}\color{black}{112122122112122121121122122112122122112112122122112}\ldots \\
\mathcal{D}^2(x) &= \color{orange}{1}\color{black}{211212211211212212211212212112112212212112212211212}\ldots \\
\mathcal{D}^3(x) &= \color{orange}{1}\color{black}{121122121121221121121221211221221121221221121121221}\ldots \\
\mathcal{D}^4(x) &= \color{orange}{2}\color{black}{122112112212112112212211212212112112212112122112112}\ldots \\
\mathcal{D}^5(x) &= \color{orange}{1}\color{black}{122122112122122112112122112112212112122122112112122}\ldots \\
\mathcal{D}^6(x) &= \color{orange}{2}\color{black}{212211212212112212211211212212112212212112112212112}\ldots \\
&\ldots
\end{aligned}
$$

Trace

$$
\begin{aligned}
\mathcal{T} : \mathcal{C}^\infty &\longrightarrow \mathcal{A}^{\mathbb{N}} \\
x &\longmapsto (\mathcal{D}^n(x)_0)_{n\geq 0}
\end{aligned}
$$

Smooth sequences

$$
\begin{aligned}
x &= 2212112112212211212212211211212212112212211212212211\ldots \\
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\mathcal{D}^5(x) &= 1122122112122122112112122112112212112122122112112122\ldots \\
\mathcal{D}^6(x) &= 2212211212212112212211211212212112212212112112212112\ldots \\
&\ldots
\end{aligned}
$$

Expansion $\mathcal{E} : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{C}^{\infty}$

$$t \longmapsto \lim_{n\to\infty} \mathfrak{p}_{t_0}\mathfrak{p}_{t_1}\ldots\mathfrak{p}_{t_{n-1}}(t_n)$$

Smooth sequences

Proposition *(Conjugacy)* [Dekking 2001] [Cassaigne, H. 2026+]

(i) $\mathcal{T}$ and $\mathcal{E}$ are **reciprocal** of each other and **continuous**.
(ii) $(\mathcal{C}^\infty, \mathcal{D})$ is **conjugated** to the full-shift $(\mathcal{A}^\mathbb{N}, S)$

$$\begin{array}{ccc}
\mathcal{C}^\infty & \xrightarrow{\mathcal{D}} & \mathcal{C}^\infty \\
\mathcal{E}\Big\uparrow\Big\downarrow\mathcal{T} & & \mathcal{E}\Big\uparrow\Big\downarrow\mathcal{T} \\
\mathcal{A}^\mathbb{N} & \xrightarrow{S} & \mathcal{A}^\mathbb{N}
\end{array}$$

Smooth sequences

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$\mathcal{E}\left((21)^\omega\right) = 2122121122122112112122112112212112112212212112\ldots$
$\mathcal{E}\left((12)^\omega\right) = 1121122122121122122112122121121122121121221121\ldots$

- $\mathcal{A} = \{a, b\}$, $a + b$ odd

Conjecture (same as κ)

Let $x \in \mathcal{C}^\infty$ over a mixed alphabet $\{a, b\}$. Then

- x is **not morphic**
- $\mathscr{L}(x) = \mathcal{C}_f^\infty$
- x is **UR** but not **LR**
- $p_x(n) = p_{\mathcal{C}_f^\infty}(n) = \Theta(n^\rho)$, $\rho = \frac{\log(a+b)}{\log\left(\frac{a+b}{2}\right)}$ [Sing 2010]
- x has **uniform frequencies**
- x has **equal letter frequencies**

Smooth words

The language $\mathcal{C}_f^\infty$ of **smooth words** is crucial

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Theorem *(Factor complexity)* [Cassaigne, H. 2026+]

$\mathcal{A} = \{a, b\}$, $\rho = \frac{\log(a+b)}{\log\left(\frac{a+b}{2}\right)}$.

(i) $\mathscr{L}(\mathcal{C}^\infty) = \mathcal{C}_f^\infty$.

(ii)

(iii)

(iv)

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(iv) If $\mathcal{A}$ is **odd**, we obtain a **new polynomial upper bound**.

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Remaining questions:

- $\mathscr{L}(x) = \mathcal{C}_f^\infty$ over mixed alphabets

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Remaining questions:

- $\mathscr{L}(x) = \mathcal{C}_f^\infty$ over mixed alphabets
- Conjectured upper bound over mixed alphabets
- Conjectured upper bound over odd alphabets

Smooth sequences over even and odd alphabets

Proposition [Sing 2002,2004,2011]

$\mathcal{A} = \{a, b\}$ **even or odd**, $\mathcal{D}(x) = x \in \mathcal{C}^\infty$.

- x is **morphic** and generated by a **primitive** substitution
- x is **LR**
- $p_x(n) = \Theta(n)$
- x has **uniform frequencies**
- x has equal **letter frequencies** if and only if $\mathcal{A}$ is even

Smooth sequences over even and odd alphabets

Theorem [Brlek, Jamet, Paquin 2008]

$\mathcal{A} = \{a, b\}$ **even**.
(i) Every smooth sequence is **recurrent**.
(ii) $\exists x \in \mathcal{C}^\infty$ s.t. $\mathscr{L}(x)$ is **not mirror-invariant**.
(iii) $\exists x \in \mathcal{C}^\infty$ s.t. $\mathscr{L}(x)$ is **not complement-invariant**.

Theorem [Brlek, Jamet, Paquin 2008]

$\mathcal{A} = \{a, b\}$ **odd**.
(i) Every smooth sequence is **recurrent**.
(ii) $\forall x \in \mathcal{C}^\infty$, $\mathscr{L}(x)$ is **mirror-invariant**.
(iii) $\exists x \in \mathcal{C}^\infty$ s.t. $\mathscr{L}(x)$ is **not complement-invariant**.

Theorem [Jamet, Marcovici, Poirier, de la Rue 2026+]

Let x be a **bi-infinite** smooth sequence over $\{1, 3\}$.
(i) x has **uniform frequencies**.
(ii) x is (almost always) **UR**.

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- **S-adic** representation of smooth sequences with two substitutions

Smooth sequences over even and odd alphabets

Proposition *(S-adic representation)* [Cassaigne, H. 2026+]

$\mathcal{A} = \{a, b\}$ **even or odd**. There exists

- a finite alphabet Σ,
- substitutions $\sigma_a, \sigma_b : \Sigma \to \Sigma^+$,
- substitutions $\pi_a, \pi_b : \Sigma \to \mathcal{A}^+$,
- a letter $A \in \Sigma$

s.t.

$$\mathcal{C}^\infty = \left\{ \lim_{n\to\infty} \pi_{t_0}\sigma_{t_1}\sigma_{t_2}\ldots\sigma_{t_n}(A^\omega) \;\middle|\; t \in \mathcal{A}^{\mathbb{N}} \right\}.$$

Moreover, the directive sequence of a smooth sequence x is $t = \mathcal{T}(x)$.

If $\mathcal{A} = \{a, b\}$ is **even**, the substitutions are

$$\sigma_a : A \longmapsto A^{\frac{a}{2}}\ B^{\frac{a}{2}} \qquad\qquad \sigma_b : A \longmapsto A^{\frac{b}{2}}\ B^{\frac{b}{2}}$$
$$B \longmapsto A^{\frac{b}{2}}\ B^{\frac{b}{2}} \qquad\qquad B \longmapsto A^{\frac{a}{2}}\ B^{\frac{a}{2}}$$

and

$$\pi_a : A \longmapsto aa \qquad\qquad \pi_b : A \longmapsto bb$$
$$B \longmapsto bb \qquad\qquad B \longmapsto aa.$$

Smooth sequences over even and odd alphabets

If $\mathcal{A} = \{a, b\}$ is **odd**, the substitutions are

$$\sigma_a : A \longmapsto A^{\frac{a-1}{2}}\ B\ C^{\frac{a-1}{2}} \qquad \sigma_b : A \longmapsto A^{\frac{b-1}{2}}\ B\ C^{\frac{b-1}{2}}$$
$$B \longmapsto A^{\frac{a-1}{2}}\ B\ C^{\frac{b-1}{2}} \qquad B \longmapsto A^{\frac{b-1}{2}}\ B\ C^{\frac{a-1}{2}}$$
$$C \longmapsto A^{\frac{b-1}{2}}\ B\ C^{\frac{b-1}{2}} \qquad C \longmapsto A^{\frac{a-1}{2}}\ B\ C^{\frac{a-1}{2}}$$

and

$$\pi_a : A \longmapsto aa \qquad \pi_b : A \longmapsto bb$$
$$B \longmapsto ab \qquad B \longmapsto ba$$
$$C \longmapsto bb \qquad C \longmapsto aa$$

We apply [Durand 2000/2003] on **proper primitive** S-adic sequences

Theorem *(Linear recurrence)* [Cassaigne, H. 2026+]

$\mathcal{A} = \{a, b\}$ **odd or even**.
(i) If $a > 1$, then every smooth sequence is **LR**.
(ii) If $a = 1$, then x is **LR** if and only if $\exists n \geq 0$ s.t.

$$(1b)^n \not\subset \mathcal{T}(x) \text{ or } S^n(\mathcal{T}(x)) = (1b)^\omega.$$

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- **LR** sequences have **uniform frequencies** and **linear factor complexity**

Smooth sequences over even and odd alphabets

Without [Durand 2000/2003], we still obtain

Theorem *(Uniform recurrence)* [Cassaigne, H. 2026+]

Over **even and odd** alphabets, every smooth sequence is **UR**.

Smooth sequences over even and odd alphabets

Theorem *(Anti-complement-invariance)* [Cassaigne, H. 2026+]

Over an **even or odd** alphabet, there exists $\mathfrak{S} \subset \mathcal{A}^*$ **finite** s.t. if $u \in \mathscr{L}(x)$ and $u \notin \mathfrak{S}$, then $\overline{u} \notin \mathscr{L}(x)$.

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The smooth sequences

$$\mathcal{E}((13)^\omega) = 1113111313111311131311131311131113131113111313 1\ldots$$

$$\mathcal{E}((31)^\omega) = 3131113131113111313111313111311131311131113131 1\ldots$$

do not contain the word 33.

Smooth sequences over even and odd alphabets

Theorem *(Factor complexity)* [Cassaigne, H. 2026+]

$\mathcal{A} = \{a, b\}$ **even**.

(i) All smooth sequences have the **same factor complexity** $p(n)$.

(ii) If $b = a + 2$, then $p(n) = \begin{cases} 1 \text{ if } n = 0 \\ 2n \text{ if } n \in [\![1, a+1]\!] \\ 2n + 2 \text{ if } n \geq a + 2 \end{cases}$.

(iii) If $b > a + 2$, then $2 < \liminf_{n\to\infty} \frac{p(n)}{n} < \limsup_{n\to\infty} \frac{p(n)}{n} < 4$.

Theorem *(Factor complexity)* [Cassaigne, H. 2026+]

$\mathcal{A} = \{a, a+2\}$ **odd** s.t. $a > 1$. Then every smooth sequence has the **same factor complexity**

$$p(n) = \begin{cases} 1 \text{ if } n = 0 \\ 2n \text{ if } n \in [\![1, a+1]\!] \\ 2n+2 \text{ if } n \geq a+2 \end{cases} .$$

Smooth sequences over even and odd alphabets

Theorem *(Factor complexity)* [Cassaigne, H. 2026+]

Let $\mathcal{A} = \{1, 3\}$.
(i) Every smooth sequence x such that $\mathcal{T}(x)$ is not eventually $(13)^\omega$ has factor complexity

$$p(n) = \begin{cases} 1 \text{ if } n = 0 \\ 2 \text{ if } n = 1 \\ 4 \text{ if } n = 2 \\ 2n + 2 \text{ if } n \geq 3 \end{cases}.$$

(ii) Every smooth sequence has linear factor complexity.
(iii) Every smooth sequence has uniform frequencies.

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(iii) Every smooth sequence has uniform frequencies.

(iii) is due to [Boshernitzan 1984]

Panorama of factor complexities

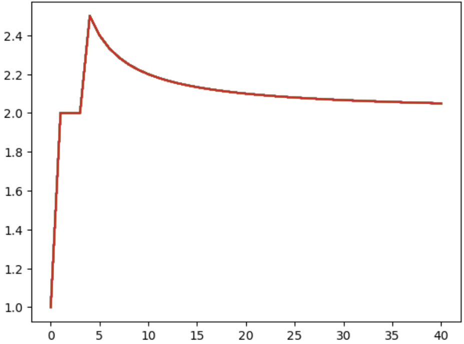

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{2, 4\}$

Panorama of factor complexities

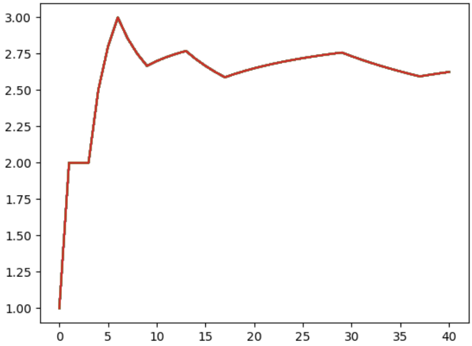

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{2, 6\}$

Panorama of factor complexities

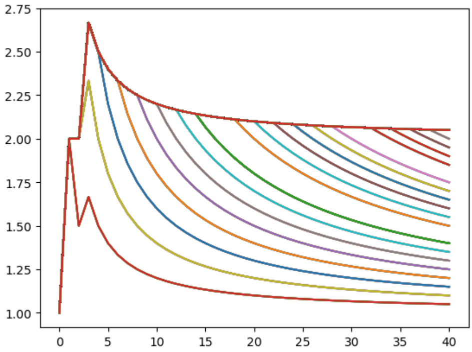

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{1, 3\}$

Panorama of factor complexities

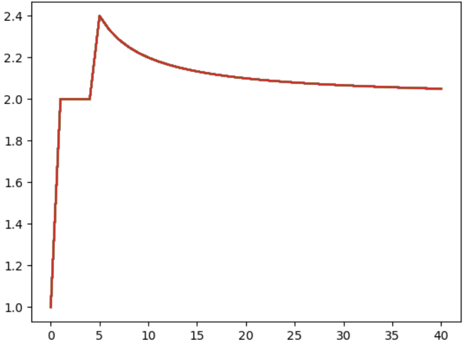

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{3, 5\}$

Panorama of factor complexities

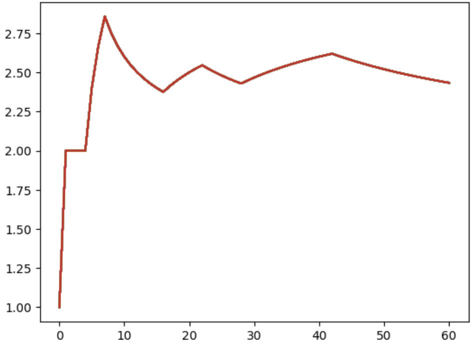

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{3, 7\}$

Panorama of factor complexities

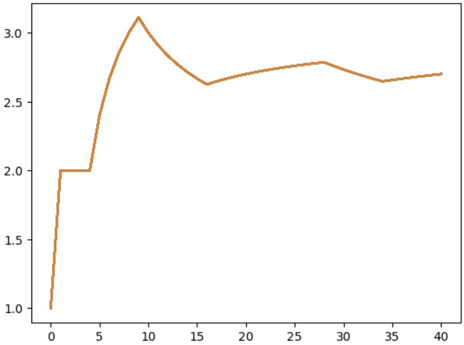

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{3, 9\}$

Panorama of factor complexities

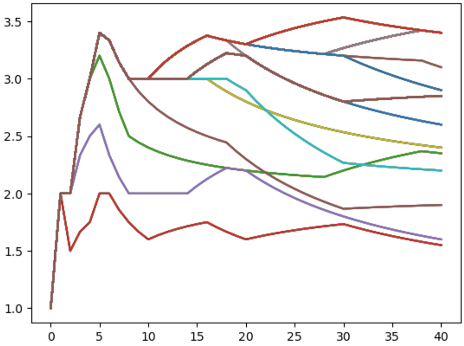

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{1,5\}$

Panorama of factor complexities

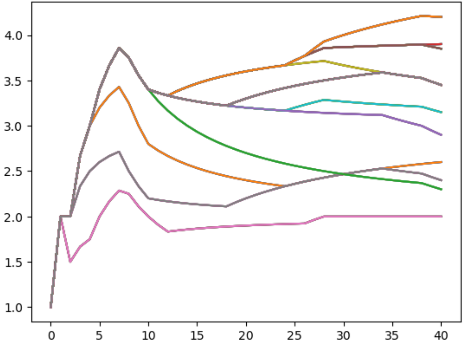

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{1, 7\}$

Panorama of factor complexities

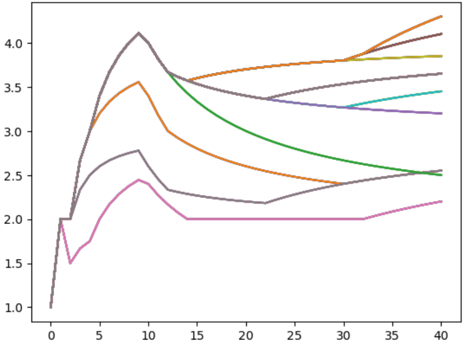

Figure: Plot of $\frac{p_x(n)}{n}$ for all $x \in \mathcal{C}^\infty$ over $\{1, 9\}$

Merci pour votre attention